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SCIENTIFIC REPORT NO. 2

"EXAMPLES OF P. D. F.'S OF SUMS AND PRODUCTS"

by

SZE-HOU CHANG and LIH-JYH WENG

Under joint support of
Contract No. AF19(604)7494 and Grant No. AF-AFOSR 62-371

SEPTEMBER 1962

NORTHEASTERN UNIVERSITY
360 HUNTINGTON AVENUE
BOSTON 15, MASSACHUSETTS

Project No. 4610
Task No. 461003

Prepared
for

ELECTRONICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
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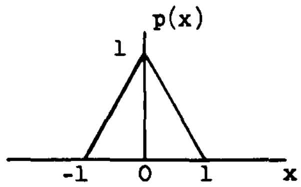
Summary

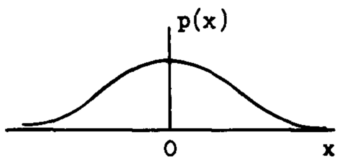
In this note the calculations of the probability density functions (p.d.f.'s) of the sums and products of random variables are demonstrated. The random variables are selected pair-wise from four variables whose probability density functions are shown below:

(a) Triangular



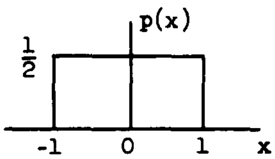
$$p(x) = \begin{cases} 0 & x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & x > 1 \end{cases} .$$

(b) Gaussian



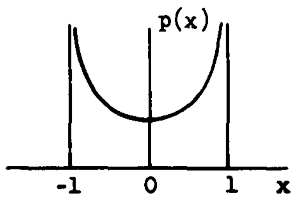
$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} .$$

(c) Rectangular



$$p(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 < x < 1 \\ 0 & x > 1 \end{cases} .$$

(d) Sinusoidal



$$p(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & x > 1 \end{cases} .$$

There are altogether 10 different pairs for the sums and the same number of pairs for the products. They are listed according to the following scheme:

	a	b	c	d
a	1	5	6	7
b		2	8	9
c			3	10
d				4

It is evident that the blank spaces represent pairs which are merely repetitions of the 10 basic pairs.

In sections I and II the p.d.f.'s are calculated respectively for the sums and products of the pairs of variables which are considered statistically independent. Then in section III, pair 2 (i.e., the gaussian-gaussian combination) is treated for the case when the two gaussian variables are dependent, as expressed by their covariance matrix. In section IV, pair 4 is treated for the case when the two sinusoidal variables are dependent.

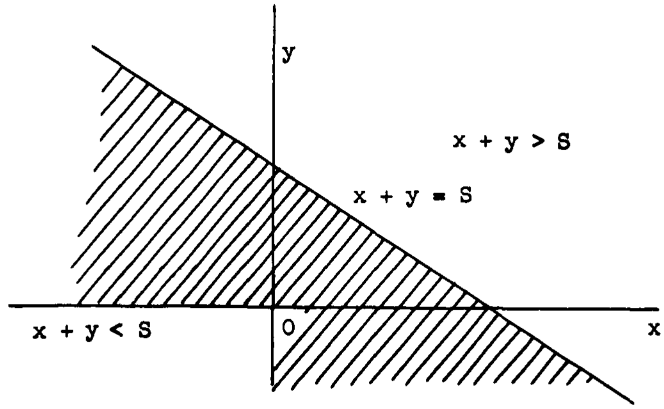
With the exception of one case, all results are expressed in terms of tabulated functions. The formulae, the tables and the graphs of the p.d.f.'s are given for all these cases.

I. SUM OF TWO INDEPENDENT VARIABLES

Let x and y be two independent random variables and s be their sum. The p.d.f.'s of x, y, and s are represented by $p_1(x)$, $p_2(y)$ and $p(s)$ respectively. The joint p.d.f. of x and y is given by

$$W_2(x, y) = p_1(x)\, p_2(y) \quad . \tag{I-1}$$

It is evident from the figure that:



$$\int_{-\infty}^{S} p(s)\, ds = \int_{-\infty}^{\infty} \int_{-\infty}^{S-x} W_2(x, y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{S-x} p_1(x)\, p_2(y)\, dy\, dx \quad . \qquad \text{(I-2)}$$

Differentiating both sides with respect to S:

$$p(S) = \int_{-\infty}^{\infty} W_2(x, S-x)\, dx \qquad \text{(I-3a)}$$

$$= \int_{-\infty}^{\infty} p_1(x)\, p_2(S-x)\, dx \quad . \qquad \text{(I-3b)}$$

Equation (I-3a) applies to the general case when x and y may or may not be dependent while equation (I-3b) applies only to the case where x and y are independent. The integral operation in (I-3b) is called the convolution of $p_1(x)$ and $p_2(y)$.

Using either (I-3a) or (I-3b), the 10 cases for the sums cited in the summary have been evaluated and the results of the evaluation are listed on the next page. Since the p.d.f.'s are all even functions of s, only the values for $s > 0$ are considered.

No.	Sum of	p(s)	
1.	a + a	$\frac{1}{6}\left[3s^3 - 6s^2 + 4\right]$	$0 < s < 1$
		$\frac{1}{6}\left[-s^3 + 6s^2 - 12s + 8\right]$	$1 < s < 2$
		0	$s > 2$
2.	b + b	$\frac{1}{2\sqrt{\pi}} e^{-\frac{s^2}{4}}$	
3.	c + c	$\frac{1}{4}(2-s)$	$0 < s < 2$
		0	$s > 2$
4.	d + d	$\frac{1}{\pi^2} K\left(\sqrt{(1 - \frac{s}{2})(1 + \frac{s}{2})}\right)$	$0 < s < 2$
		where K is the complete elliptic integral of the first kind.	
		0	$s > 2$
5.	a + b	$(s+1)\, \phi^{-1}(s+1) - 2s\, \phi^{-1}(s) + (s-1)\, \phi^{-1}(s-1)$ $+ \frac{\sqrt{2}}{\sqrt{\pi}}\left[e^{-\frac{s^2}{2}}\left(e^{-\frac{1}{2}} \cosh s-1\right)\right]$,	$0 < s < 2$
		where ϕ^{-1} is the error integral.	
		0	$s > 2$
6.	a + c	$\frac{1}{4}(2-s^2)$	$0 < s < 1$
		$\frac{1}{4}(s^2 - 4s + 4)$	$1 < s < 2$
		0	$s > 2$
7.	a + d	$\frac{1}{\pi}\left\{(s+1)\cos^{-1} s + (1-s)\left[\sin^{-1} s + \sin^{-1}(1-s)\right] + \sqrt{2s-s^2} - 2\sqrt{1-s^2}\right\}$	$0 < s < 1$
		$\frac{1}{\pi}\left\{(1-s)\cos^{-1}(s-1) + \sqrt{2s-s^2}\right\}$	$1 < s < 2$

No.	Sum of	p(s)
8.	b + c	$\frac{1}{2}\left[\phi^{-1}(s+1) - \phi^{-1}(s-1)\right]$
9.	b + d	$\frac{1}{\sqrt{2}\,\pi^{\frac{3}{2}}} \int_{s-1}^{s+1} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{1-(s-x)^2}}\,dx$
10.	c + d	$\begin{cases} \frac{1}{2\pi}\cos^{-1}(s-1) & 0 < s < 2 \\ 0 & s > 2 \end{cases}$

The graphs of these p.d.f.'s are shown as Figs. 1-10. With the exception of case 9, all graphs were obtained from tabulated functions. Case 9 was obtained by direct computation* using IBM Computer 1620.

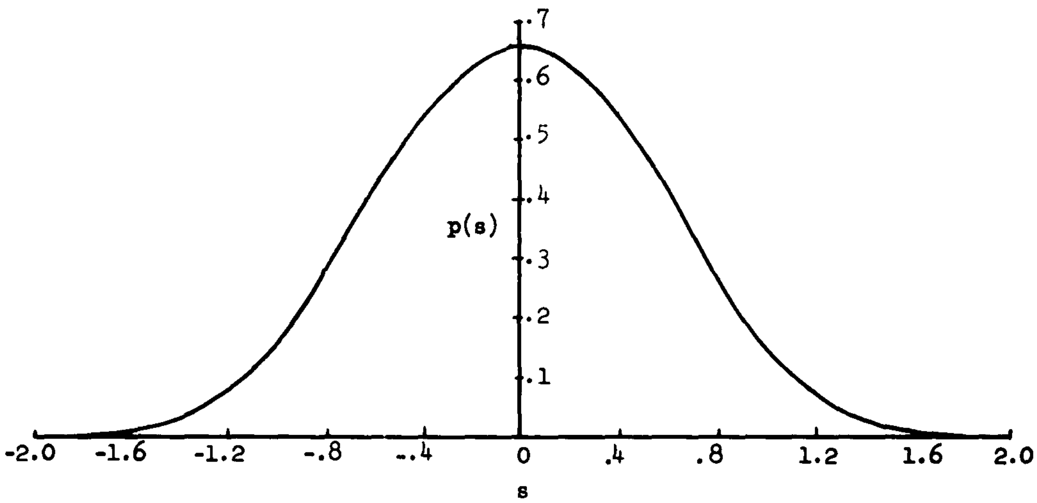


Fig. 1. P.D.F. of $s = x + y$, x and y independent, x = triangular variate, y = triangular variate.

*W.H. Lob helped in programing this computation.

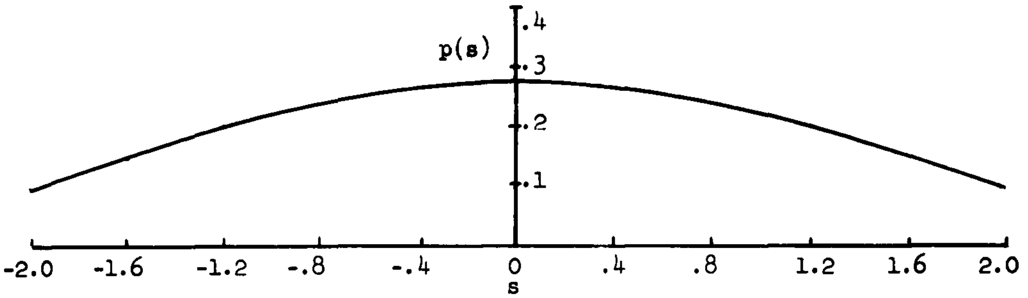


Fig. 2. P.D.F. of s = x + y, x and y independent,
x = gaussian variate, y = gaussian variate.

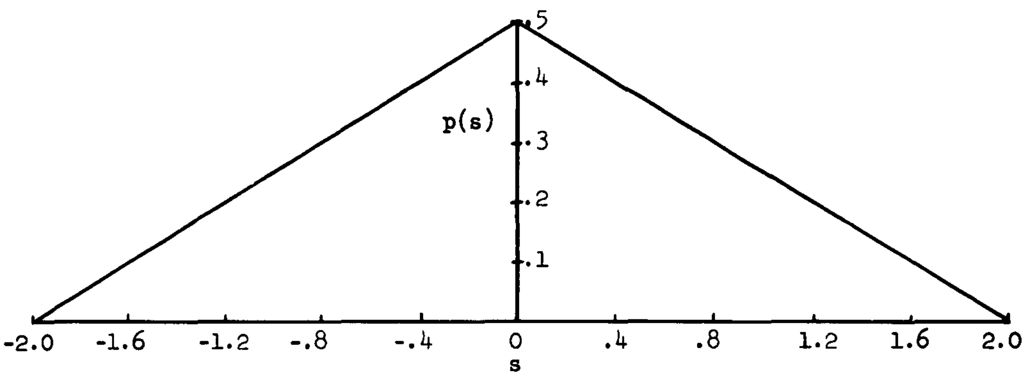


Fig. 3. P.D.F. of s = x + y, x and y independent,
x = rectangular variate, y = rectangular variate.

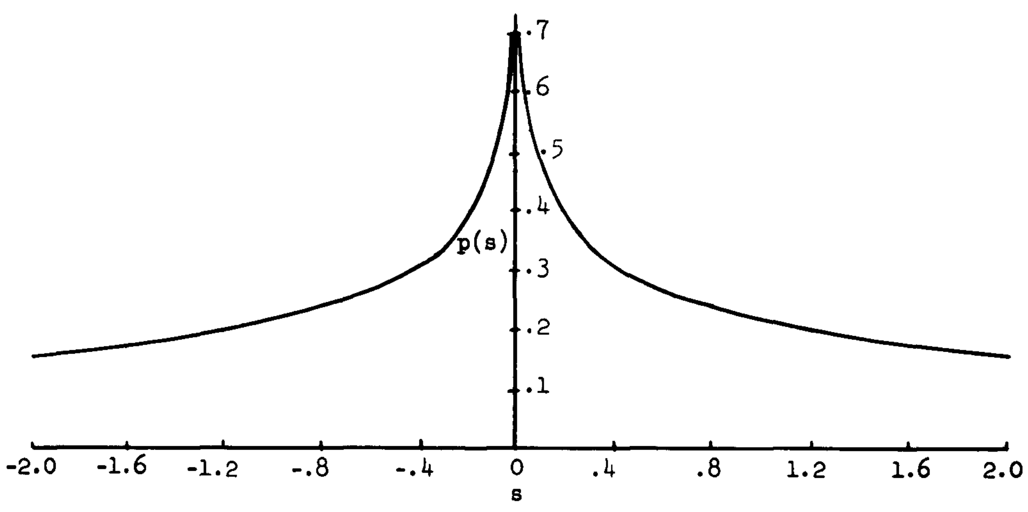


Fig. 4. P.D.F. of s = x + y, x and y independent,
x = sinusoidal variate, y = sinusoidal variate.

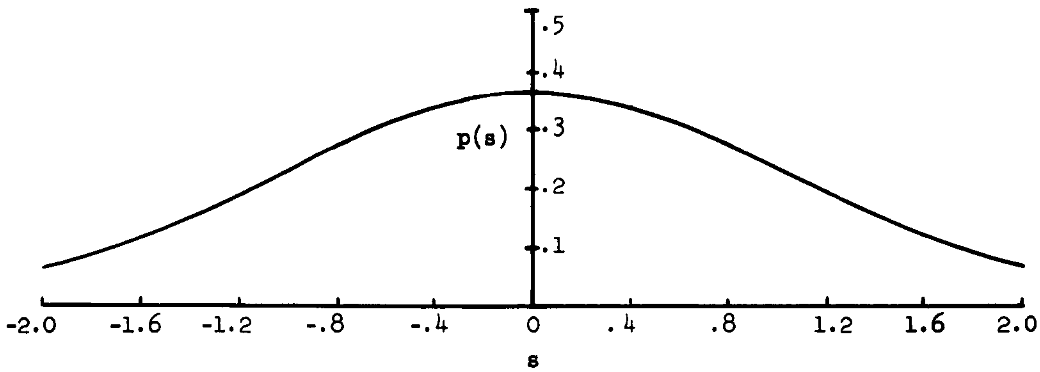


Fig. 5. P.D.F. of $s = x + y$, x and y independent,
x = triangular variate, y = gaussian variate.

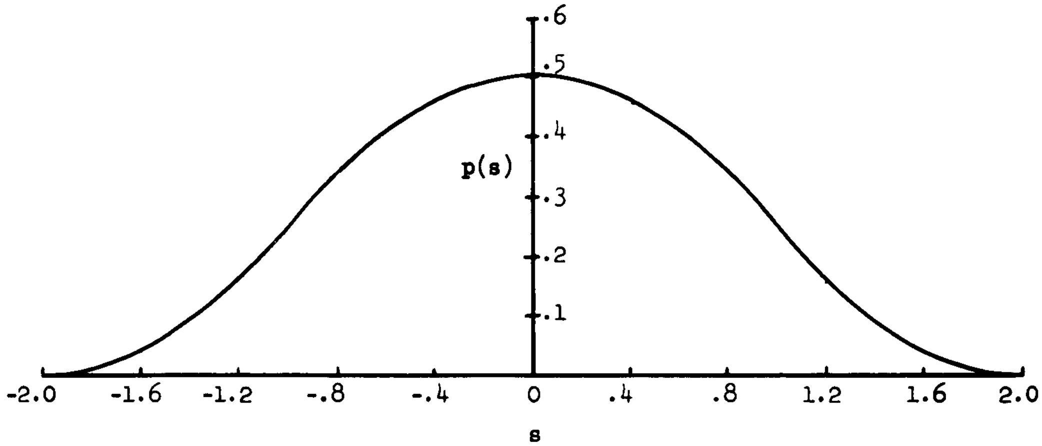


Fig. 6. P.D.F. of $s = x + y$, x and y independent,
x = triangular variate, y = rectangular variate.

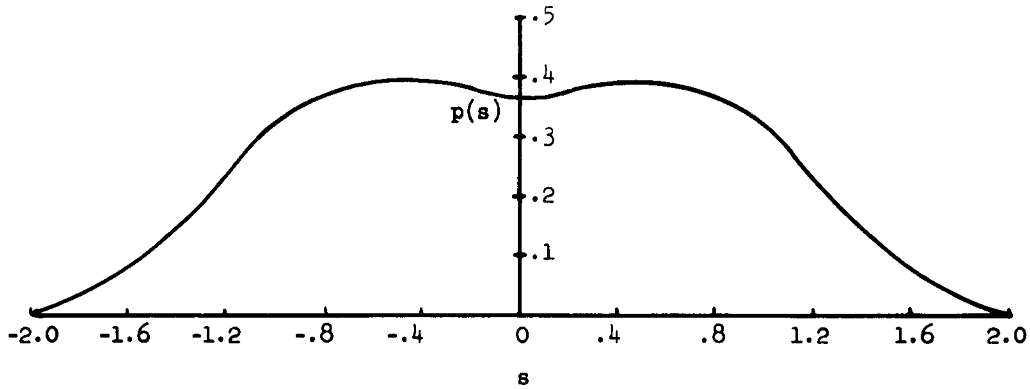


Fig. 7. P.D.F. of $s = x + y$, x and y independent,
x = triangular variate, y = sinusoidal variate.

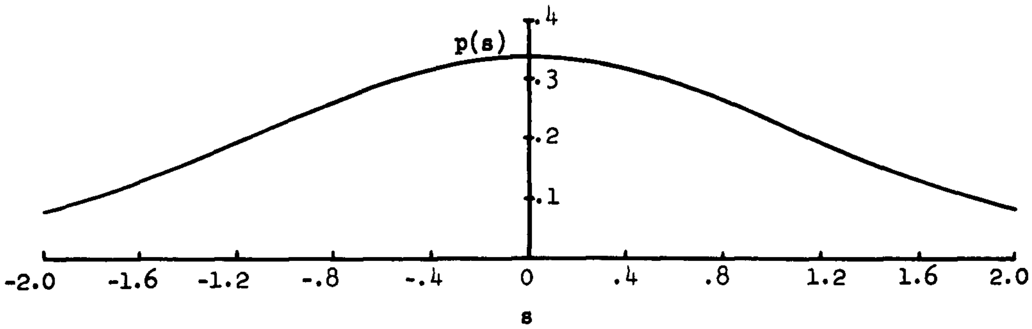


Fig. 8. P.D.F. of s = x + y, x and y independent,
x = gaussian variate, y = rectangular variate.

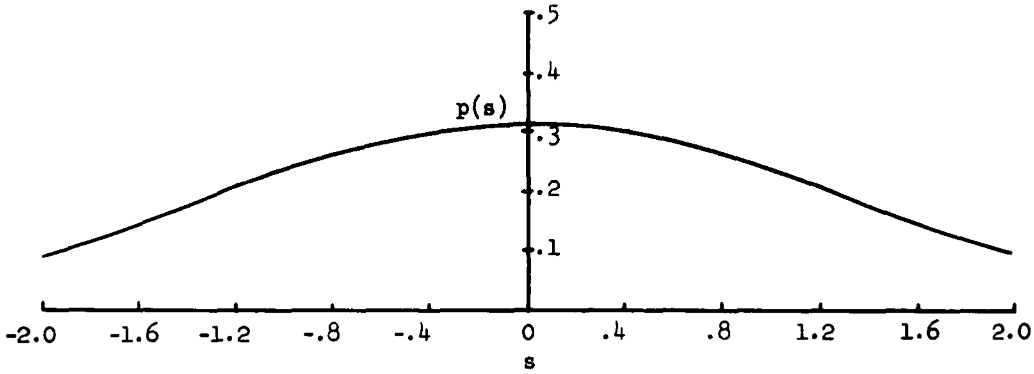


Fig. 9. P.D.F. of s = x + y, x and y independent,
x = gaussian variate, y = sinusoidal variate.

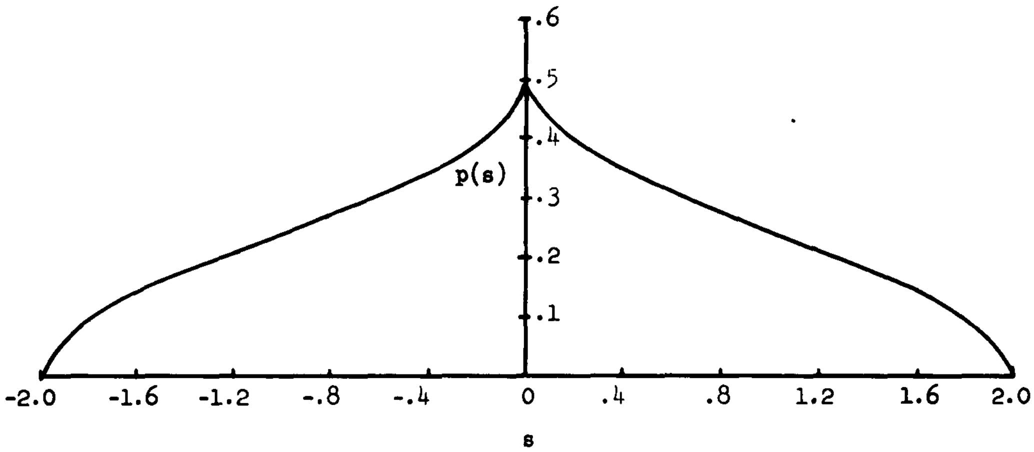


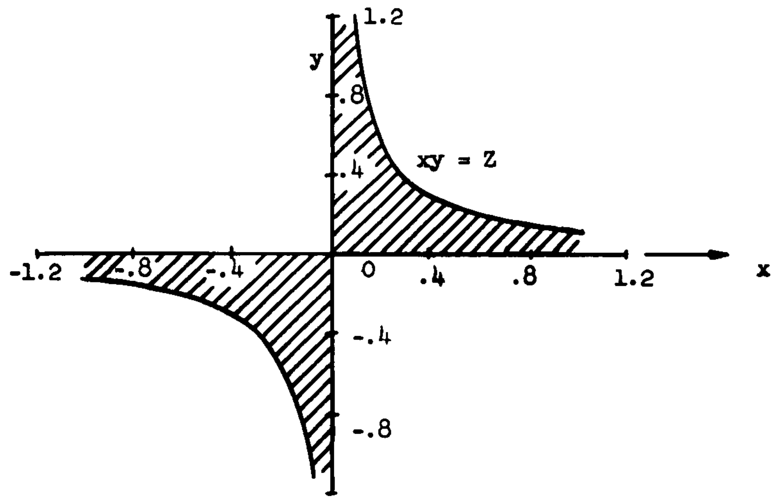
Fig. 10. P.D.F. of s = x + y, x and y independent,
x = rectangular variate, y = sinusoidal variate.

II. PRODUCT OF TWO INDEPENDENT VARIABLES

Let x and y be two independent random variables and z be their product. The p.d.f. of x, y and z are represented by $p_1(x)$, $p_2(y)$ and $p(z)$ respectively. The joint p.d.f. of x and y is given by

$$W_2(x, y) = p_1(x)\, p_2(y) \quad . \tag{II-1}$$

It is evident from the figure that for $Z > 0$



$$\int_0^Z p(z)\, dz = 2\int_0^\infty \int_0^{\frac{Z}{x}} W_2(x, y)\, dy\, dx$$

$$= 2\int_0^\infty \int_0^{\frac{Z}{x}} p_1(x)\, p_2(y)\, dy\, dx \quad . \tag{II-2}$$

Differentiating both sides with respect to Z,

$$p(Z) = 2\int_0^\infty W_2(x, \tfrac{Z}{x})\, \tfrac{1}{x}\, dx \tag{II-3a}$$

$$= 2\int_0^\infty p_1(x)\, p_2(\, \tfrac{Z}{x}\,)\, \tfrac{1}{x}\, dx \quad . \tag{II-3b}$$

Equations (II-3b) applies only to the case where x and y are independent while (II-3a) is not so restricted.

Using either (II-3a) or (II-3b), the 10 cases for the products cited in the summary have been evaluated and the results of the evaluation are listed below. Again, since the p.d.f.'s are even functions of z, only positive values of z are considered.

No.	Product of	p(z)	
1.	a × a	$2\left[(1+z)\ \ln\frac{1}{z} - 2(1-z)\right]$	$0 < z < 1$
		0	$z > 1$
2.	b × b	$\frac{1}{2}\, i\, H_o^{(1)}\,(iz)$ where $H_o^{(1)}$ is the Hankel's function of zero order.	
3.	c × c	$\frac{1}{2}\ln\frac{1}{z}$	$0 < z < 1$
		0	$z > 1$
4.	d × d	$\frac{1}{\pi^2}\,K(1-z^2)$ where K is the complete elliptic integral of the first kind.	$0 < z < 1$
		0	$z > 1$
5.	a × b	$\frac{1}{\sqrt{2\pi}}\,Ei\left(\frac{z^2}{2}\right) - 2\phi^0(z) + 2z\left[\frac{1}{2} - \phi^{-1}(z)\right]$ where Ei is the exponential integral, ϕ^0 the gaussian p.d.f., and ϕ^{-1} is the error integral.	
6.	a × c	$\ln\frac{1}{z} - (1-z)$	$0 < z < 1$
		0	$z > 1$
7.	a × d	$\frac{2}{\pi}\left[\ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}\right]$	$0 < z < 1$
		0	$z > 1$
8.	b × c	$-\frac{1}{2\sqrt{2\pi}}\,Ei\left(-\frac{z^2}{2}\right)$	

No.	Product of	p(z)	
9.	b × d	$\frac{1}{2}\,\phi^{0}\left(\frac{z}{\sqrt{2}}\right)\left[i\,H_0^{(1)}\left(i\,\frac{z^2}{4}\right)\right]$ where $H_0^{(1)}$ is the Hankel's function of zero order.	
10.	c × d	$\frac{1}{\pi}\ln\left[\frac{1+\sqrt{1-z^2}}{z}\right]$	$0 < z < 1$
		0	$z > 1$

The graphs of these 10 p.d.f.'s are shown as Figs. 11-20.

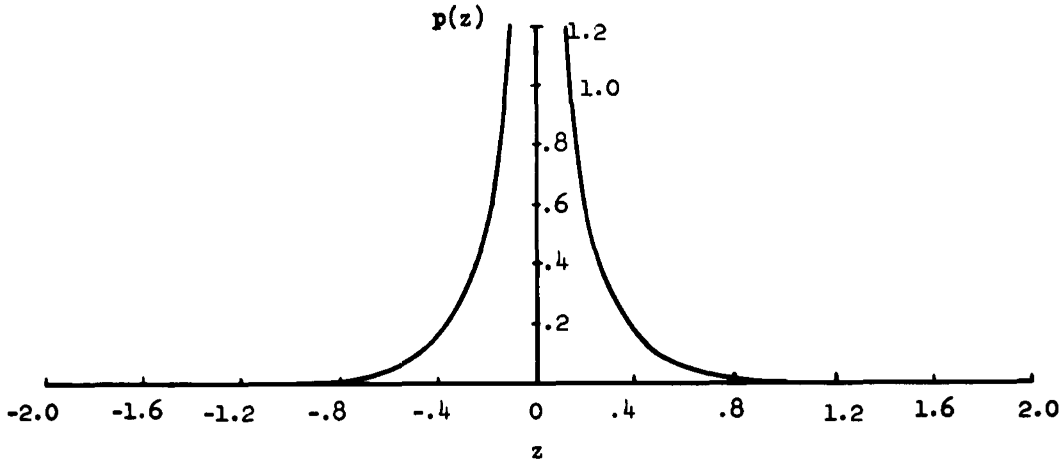


Fig. 11. P.D.F. of $z = xy$, x and y independent, x = triangular variate, y = triangular variate.

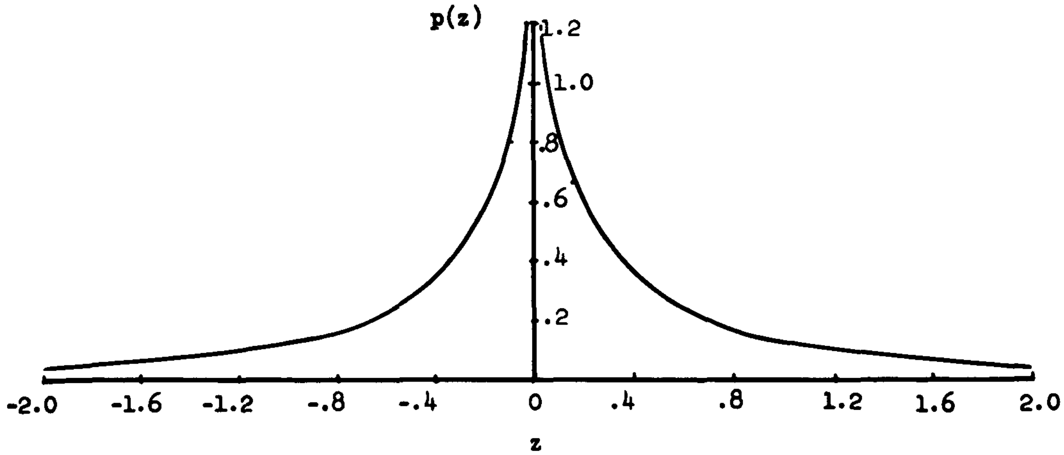


Fig. 12. P.D.F. of $z = xy$, x and y independent, x = gaussian variate, y = gaussian variate.

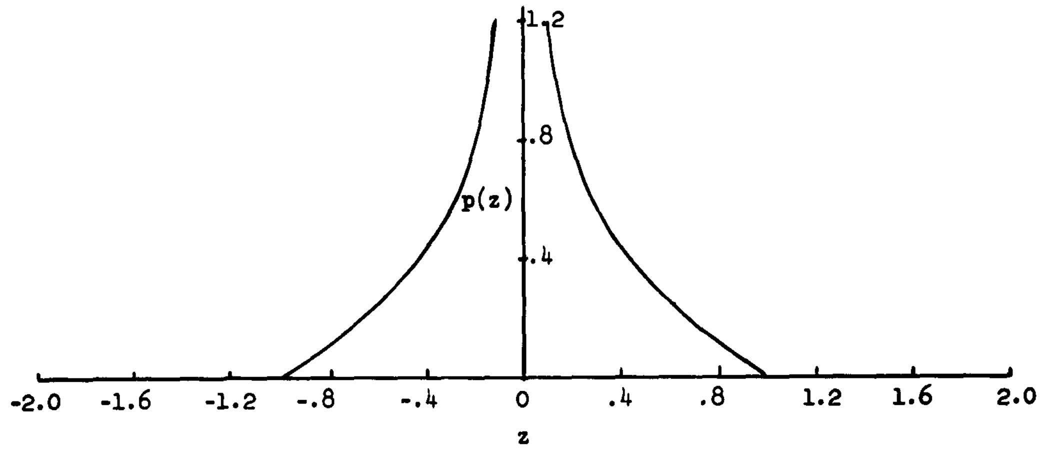


Fig. 13. P.D.F. of z = xy, x and y independent,
x = rectangular variate, y = rectangular variate.

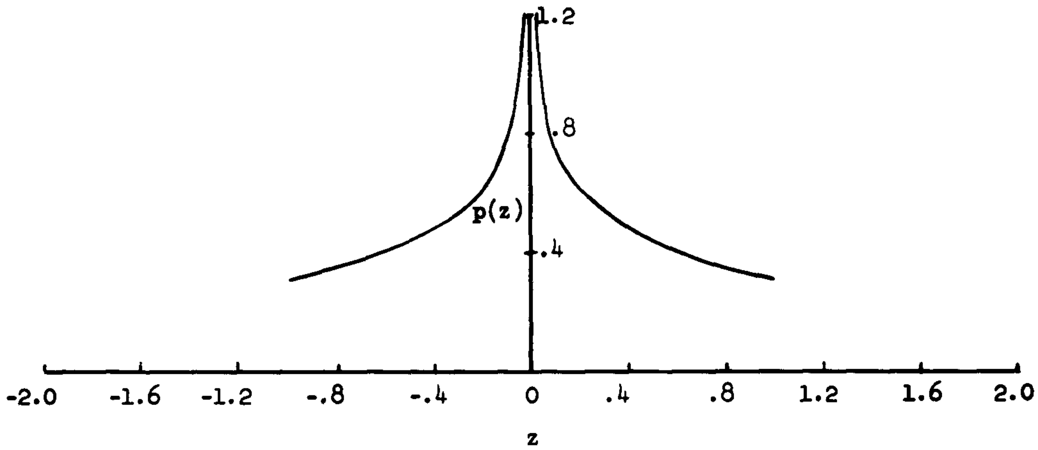


Fig. 14. P.D.F. of z = xy, x and y independent,
x = sinusoidal variate, y = sinusoidal variate.

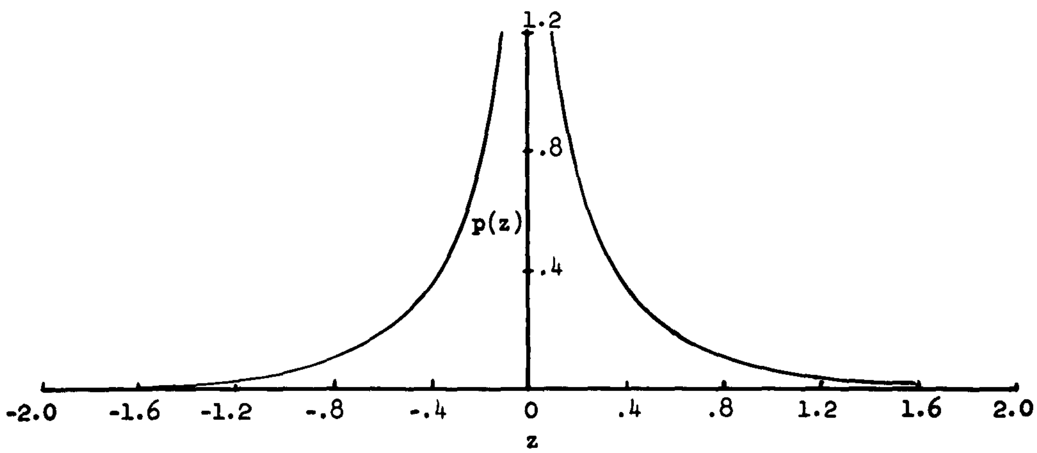


Fig. 15. P.D.F. of z = xy, x and y independent,
x = triangular variate, y = gaussian variate.

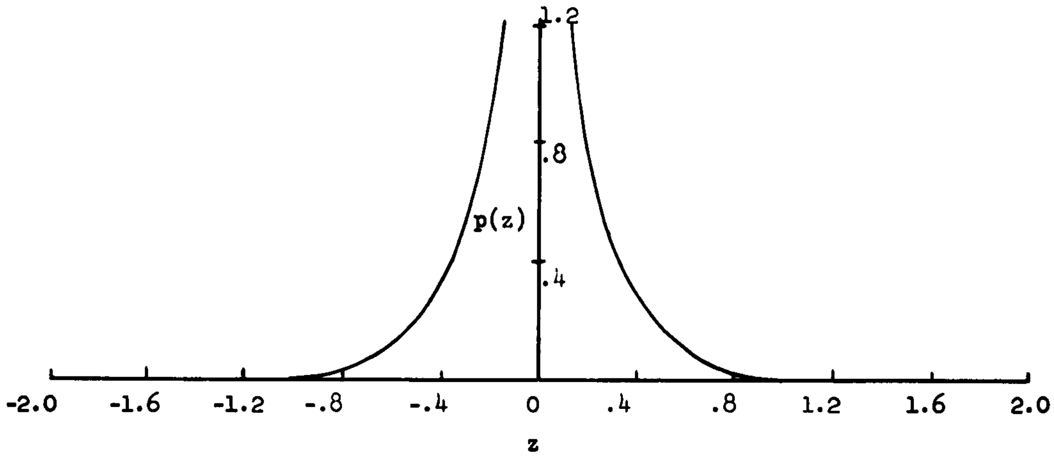


Fig. 16. P.D.F. of z = xy, x and y independent, x = triangular variate, y = rectangular variate.

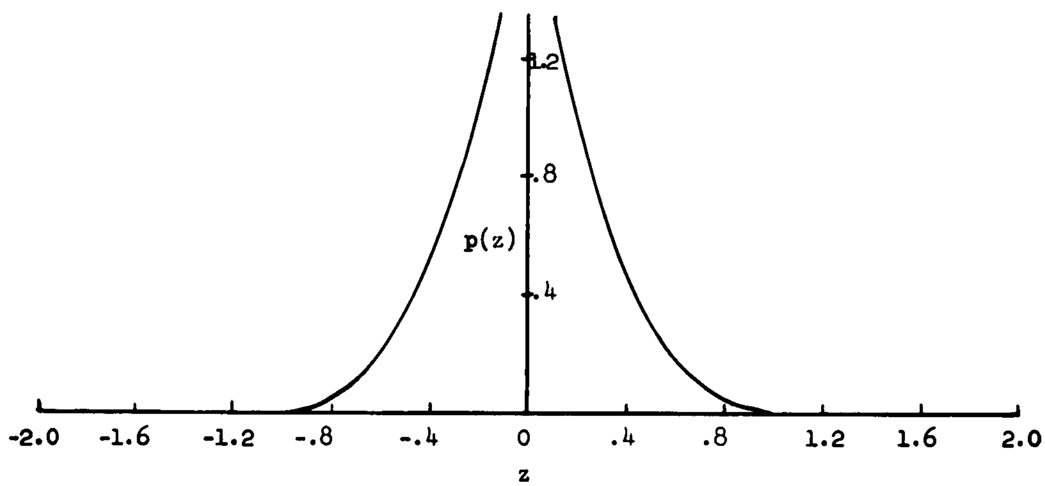


Fig. 17. P.D.F. of z = xy, x and y independent, x = triangular variate, y = sinusoidal variate.

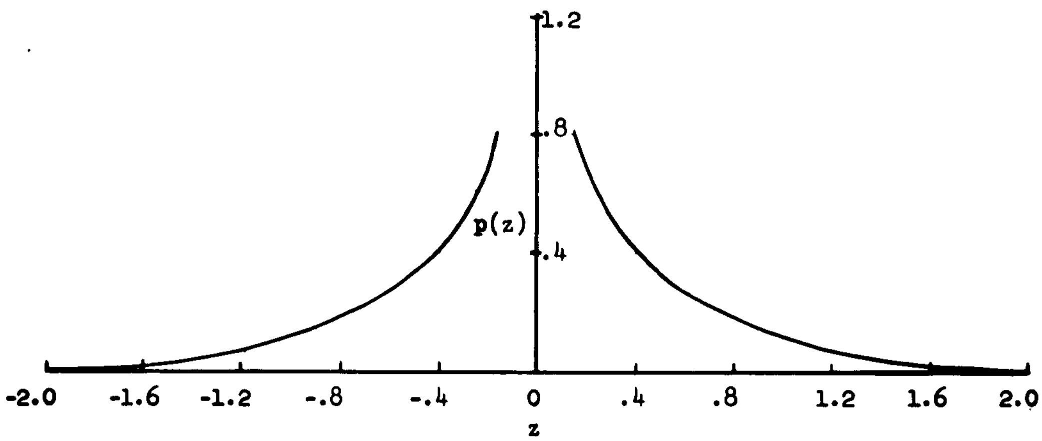


Fig. 18. P.D.F. of z = xy, x and y independent x = gaussian variate, y = rectangular variate

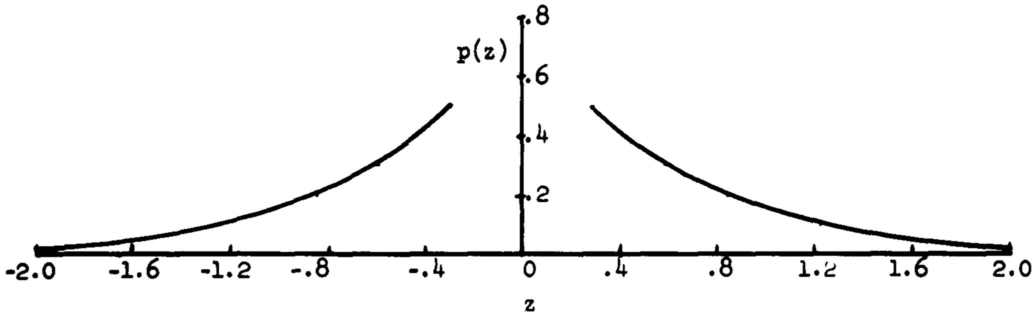


Fig. 19. P.D.F. of z = xy, x and y independent,
x = gaussian variate, y = sinusoidal variate.

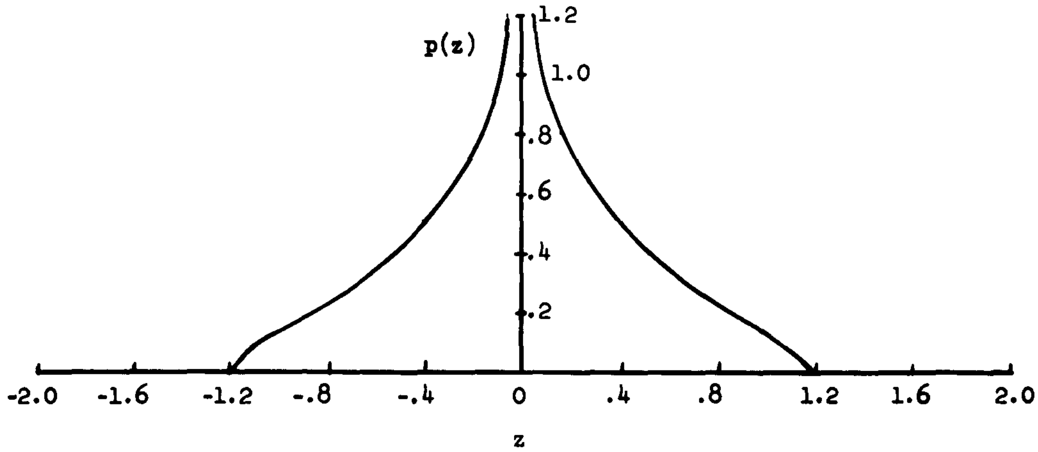


Fig. 20. P.D.F. of z = xy, x and y independent,
x = rectangular variate, y = sinusoidal variate.

The first four p.d.f.'s of z are plotted in the same sheet, Fig. 21. For a more rational comparison, the variances of the four original p.d.f.'s of x have all been normalized to unity. In so doing, the scale for the gaussian variable is kept the same while the ranges of the triangular, rectangular and sinusoidal variables are extended from (-1, 1) to $(-\sqrt{6}, \sqrt{6})$, $(-\sqrt{3}, \sqrt{3})$, and $(-\sqrt{2}, \sqrt{2})$ respectively.

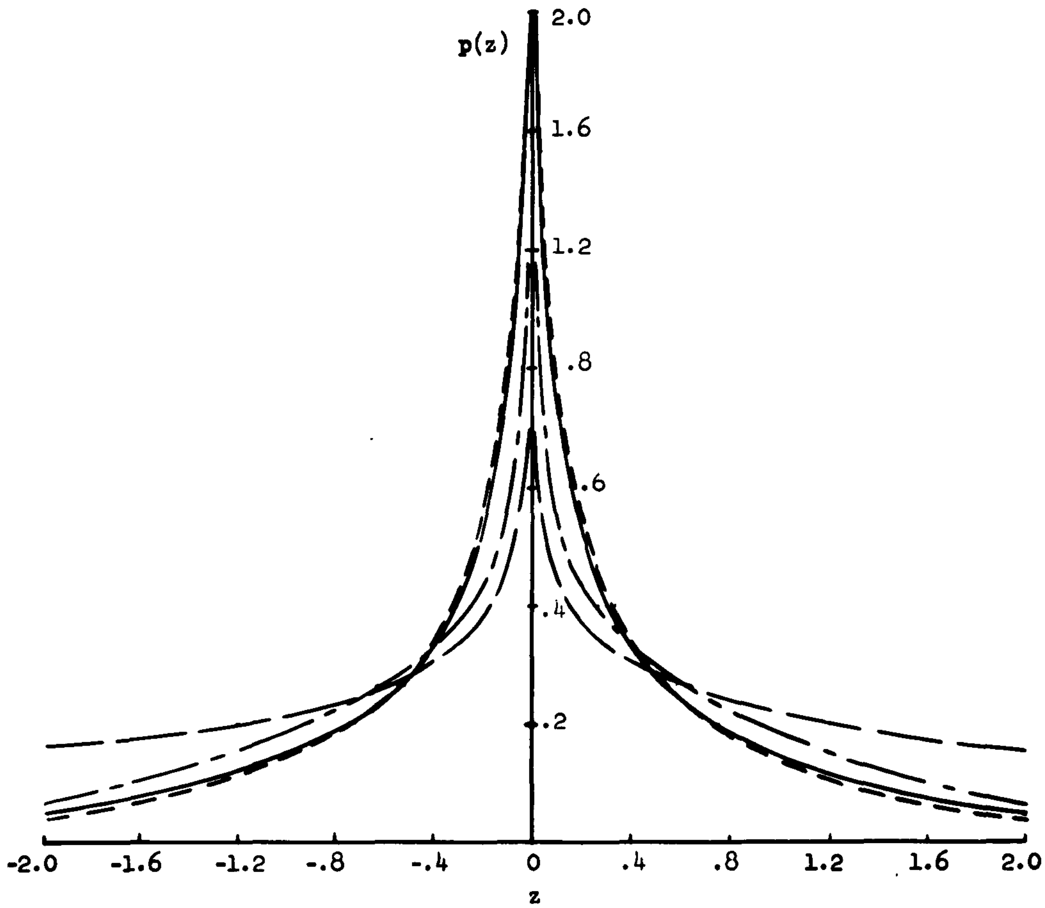


Fig. 21. P.D.F.'s of Products of

(a) — — — — Two independent gaussian variates

(b) ———— Two independent triangular variates

(c) —-—-— Two independent rectangular variates

(d) —— —— Two independent sinusoidal variates

Normalized variance = 1

III. SUM AND PRODUCT OF TWO DEPENDENT GAUSSIAN VARIABLES

Let x and y be two dependent gaussian variables whose joint p.d.f. is

$$W_2(x, y) = \frac{1}{2\pi} \frac{1}{\sqrt{|b|}} e^{-\frac{Q^{-1}(x, y)}{2}} \qquad \text{(III-1)}$$

where the covariance matrix is

$$[b] = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} , \qquad \text{(III-2)}$$

its determinant is

$$|b| = 1-\rho^2 \quad , \qquad \text{(III-3)}$$

and the quadratic form is

$$Q^{-1}(x, y) = \frac{1}{|b|}\left[x^2 - 2\rho xy + y^2\right] \quad . \qquad \text{(III-4)}$$

It is required in this section to find the p.d.f.'s of the sum $s = x + y$ and of the product $z = xy$.

Sum

First, we determine p(s). From equation (I-3a) and (III-1)

$$p(s) = \int_{-\infty}^{\infty} W_2(x, s-x)\, dx$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}[x^2 - 2\rho x(z-x) + (z-x)^2]}\, dx \quad .$$

Completing the square in the exponent and simplifying

$$p(s) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{z^2}{4(1+\rho)}} \int_{-\infty}^{\infty} e^{-\frac{1}{1-\rho}(x - \frac{z}{2})^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\ \sqrt{2(1+\rho)}} e^{-\frac{z^2}{2.2(1+\rho)}} \left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{1-\rho}{2}}} \int_{-\infty}^{\infty} e^{-\frac{(x - \frac{z}{2})^2}{2\,(\frac{1-\rho}{2})}}\, dx\right] \quad .$$

The last factor in the bracket is equal to unity, therefore

$$p(s) = \frac{1}{\sqrt{2\pi}\sqrt{2(1+\rho)}} e^{-\frac{z^2}{2.2(1+\rho)}} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2\sigma^2}} , \qquad \text{(III-5)}$$

where

$$\sigma^2 = 2(1+\rho)$$

is the variance of s.

Figure 22 shows p(s) for four values of the correlation coefficient ρ (here ρ is numerically equal to the covariance of x and y). When $\rho = -1$ the variables x and y are entirely dependent and in fact $y = -x$ with probability one. Therefore, p(s) is a delta function at $s = 0$. When $\rho = 0$, the variables x and y are independent*. This particular p(s) is the same as the case 2 of section I, where the variance of s is 2. When $\rho = 1$, the variables x and y are again entirely dependent but this time $y = x$ with probability one. Therefore p(s), while still gaussian in shape, has $\sigma = 2$ and variance $\sigma^2 = 4$.

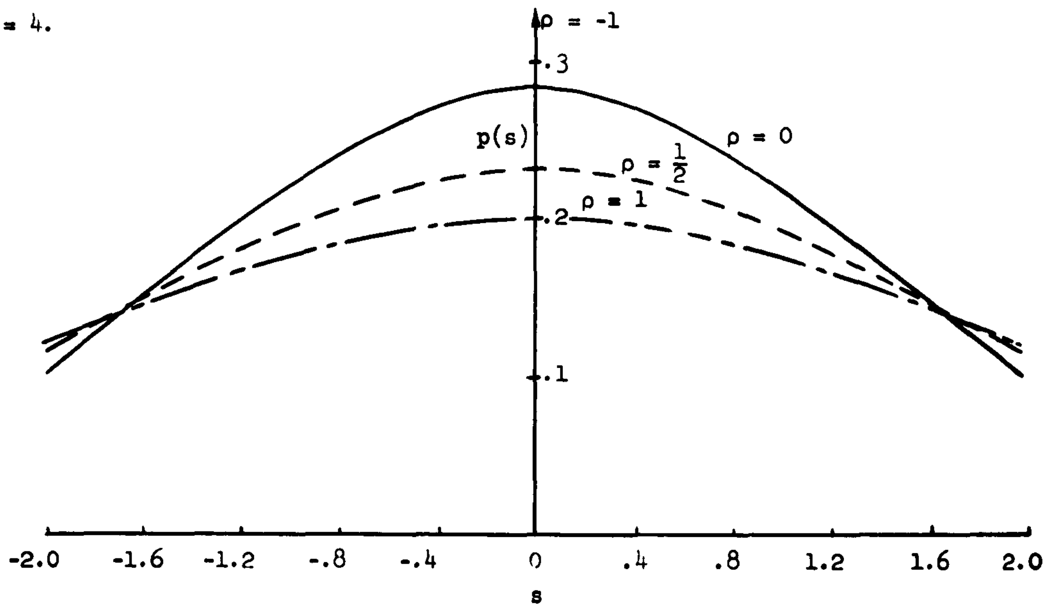


Fig. 22. P.D.F. of the Sum of 2 Dependent Gaussian Variates.

*Note that it is a property of gaussian variables that zero correlation implies independence of variables and vice versa.

Product

Next, we determine $p(z)$. From equations (II-3a) and (III-1), we have, for $z > 0$

$$p(z) = 2\int_0^\infty W_2(x, \tfrac{z}{x})\, \tfrac{1}{x}\, dx$$

$$= \frac{1}{\pi\sqrt{1-\rho^2}} \int_0^\infty e^{-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho z + \frac{z^2}{x^2}\right)} \frac{1}{x}\, dx$$

$$= \frac{1}{\pi\sqrt{1-\rho^2}}\, e^{\frac{\rho z}{1-\rho^2}} \int_0^\infty e^{-\frac{1}{2(1-\rho^2)}\left(x^2 + \frac{z^2}{x^2}\right)} \frac{1}{x}\, dx$$

$$= \frac{1}{\pi\sqrt{1-\rho^2}}\, e^{\frac{\rho z}{1-\rho^2}} \int_0^\infty e^{-\frac{z}{2\sqrt{1-\rho^2}}\left[\left(\frac{x}{\sqrt{z}\,\sqrt[4]{1-\rho^2}}\right)^2 + \left(\frac{\sqrt{z}\,\sqrt[4]{1-\rho^2}}{x^2}\right)^2\right]} \frac{1}{x}\, dx \quad .$$

Let

$$\frac{x}{\sqrt{z}\,\sqrt[4]{1-\rho^2}} = e^\theta \quad .$$

Then the p.d.f. becomes

$$p(z) = \frac{1}{\pi\sqrt{1-\rho^2}}\, e^{\frac{\rho z}{1-\rho^2}} \int_{-\infty}^\infty e^{-\frac{z}{\sqrt{1-\rho^2}}\cosh 2\theta}\, d\theta$$

$$= \frac{1}{2\sqrt{1-\rho^2}}\, e^{\frac{\rho z}{1-\rho^2}} \left[i\, H_0^{(1)}\left(i\, \frac{|z|}{1-\rho^2}\right)\right] \quad . \tag{III-6}$$

The last step is given on p. 479, Courant-Hilbert, "Methods of Mathematical Physics", volume I. $H_0^{(1)}$ is the Hankel's function, zero order, first kind, with imaginary argument. It is tabulated on pp. 236-242, Jahnke and Emde, "Table of Functions", 4th edition. A similar integral was evaluated by this method in case 2 of section II.

For $z < 0$ it can be shown that the formula for $p(z)$ remains almost the same except for a change in the sign of the argument of the function $H_0^{(1)}$. Therefore,

$$p(z) = \frac{1}{2\sqrt{1-\rho^2}}\, e^{\frac{\rho z}{1-\rho^2}} \left[i\, H_0^{(1)}\left(i\, \frac{|z|}{1-\rho^2}\right)\right] \quad . \tag{III-7}$$

Figure 23 shows p(z) for three values of ρ. When $\rho = 1$, $z = x^2$, and p(z) is reduced to the simpler expression derivable by a more direct method,

$$p(z) = p(x^2) = \begin{cases} 0 & z < 0 \\ \dfrac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} & z > 0 \end{cases} \quad . \tag{III-8}$$

When

$$\rho = -1, \quad z = -x^2 \quad ,$$

and

$$p(z) = p(-x^2) = \begin{cases} \dfrac{1}{\sqrt{-2\pi z}} e^{+\frac{z}{2}} & z < 0 \\ 0 & z > 0 \end{cases} \quad . \tag{III-9}$$

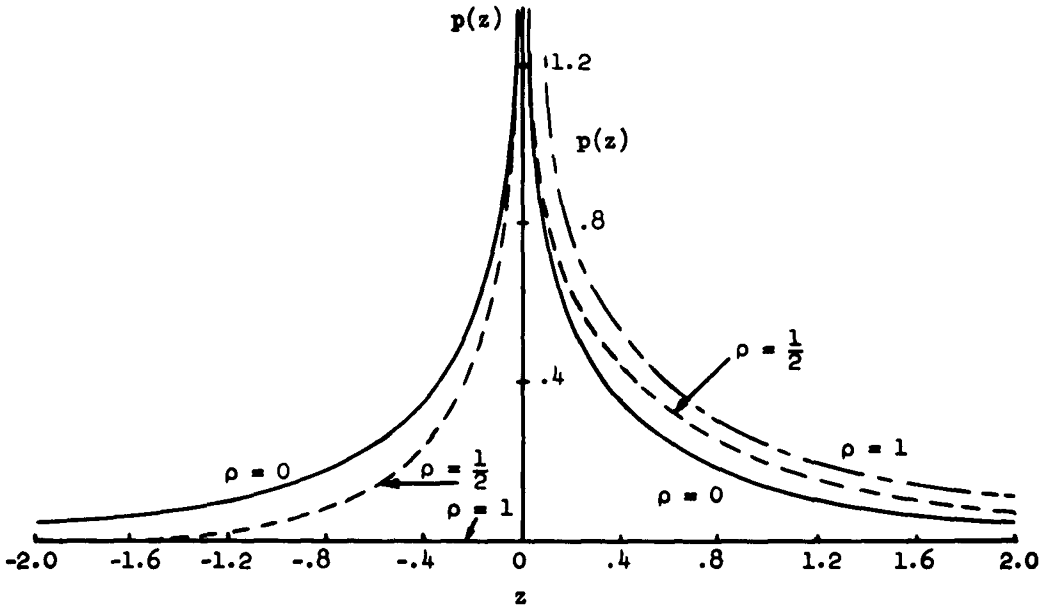


Fig. 23. P.D.F. of the Product of 2 Dependent Gaussian Variates.

When $\rho = 0$, x and y are independent. The p(z) corresponding to this particular case is the same as the case 2 of section II, namely

$$p(z) = \frac{1}{2}\left[i\, H_0^{(1)}\,(iz)\right] \quad . \tag{III-10}$$

IV. SUM AND PRODUCT OF TWO DEPENDENT INVERSE-SINUSOIDAL VARIABLES

For the variables other than gaussian, there is no longer a one-to-one relation between statistical dependence and correlation. While it is true that independent variables are not correlated, zero (linear) correlation does not always insure independence. In other words, the covariance matrix of two non-gaussian variables does not specify uniquely the joint p.d.f.

In order to demonstrate the calculation of p.d.f.'s of the sum and product of two dependent sinusoidal variables, it is desirable to select a suitable dependence condition so that the joint p.d.f. will resemble to a certain degree that of the gaussian variables. It is well known that the constant-level contours of the joint p.d.f. of two gaussian variables are ellipses. In the following example, the boundary of the joint p.d.f. of two inverse-sinusoidal variables is also an ellipse.

Let θ and α be two independent random variables whose p.d.f.'s $p_1(\theta)$ and $p_2(\alpha)$ are both assumed to be rectangular and centered about zero. However, the range of θ is assumed to be $(-\pi, \pi)$ while that of α is $(-\Delta, \Delta)$ where $|\Delta| \leq \pi$. Although θ and α are independent, the derived variables $x = \sin\theta$ and $y = \sin(\theta + \alpha)$ will be statistically dependent. We shall evaluate the p.d.f.'s of the sum $s = x + y$ and the product $z = xy$. But first, it is required to derive the expression for the joint p.d.f. $W_2(x, y)$.

It is given that

$$p_1(\theta) = \frac{1}{\pi} \text{rect}\left(\frac{\theta}{\pi}\right)^{*} \qquad \text{(IV-1)}$$

$$p_2(\alpha) = \frac{1}{2\Delta} \text{rect}\left(\frac{\alpha}{2\Delta}\right) \quad . \qquad \text{(IV-2)}$$

The joint p.d.f. of θ and α is

$$W_2(\theta, \alpha) = p_1(\theta)\; p_2(\alpha) = \frac{1}{2\Delta\pi} \text{rect}\left(\frac{\theta}{\pi}\right) \text{rect}\left(\frac{\alpha}{2\Delta}\right) \quad . \qquad \text{(IV-3)}$$

By the standard technique of change of variables, the joint p.d.f. of (x, y) is given by

$$W_2(x, y) = W_2(\theta, \alpha)\, \frac{1}{J} \qquad \text{(IV-4)}$$

where J is the jacobian of the transformation, given as follows

$$J\left(\frac{x,\ y}{\theta,\ \alpha}\right) = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} \end{vmatrix} = \begin{vmatrix} \cos\theta & \cos(\theta + \alpha) \\ 0 & \cos(\theta + \alpha) \end{vmatrix} = \cos\theta \cos(\theta + \alpha)$$

$$= \sqrt{1-x^2}\ \sqrt{1-y^2} \quad . \qquad \text{(IV-5)}$$

---

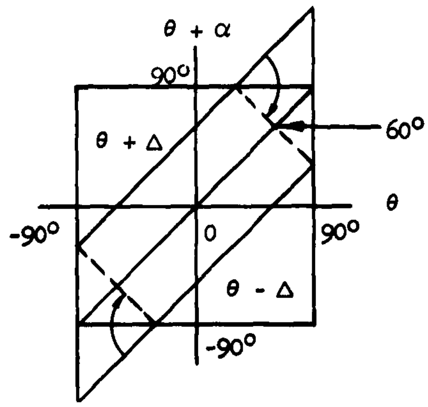
*The function $\text{rect}\left(\frac{\theta}{\pi}\right) = \begin{cases} 0 & \theta < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0 & \theta > \frac{\pi}{2} \end{cases}$.

It can be shown that due to periodicity of the sine function, the results of assuming either $p_1(\theta) = \frac{1}{\pi} \text{rect}\left(\frac{\theta}{\pi}\right)$ or $p_1(\theta) = \frac{1}{2\pi} \text{rect}\left(\frac{\theta}{2\pi}\right)$ will be the same.

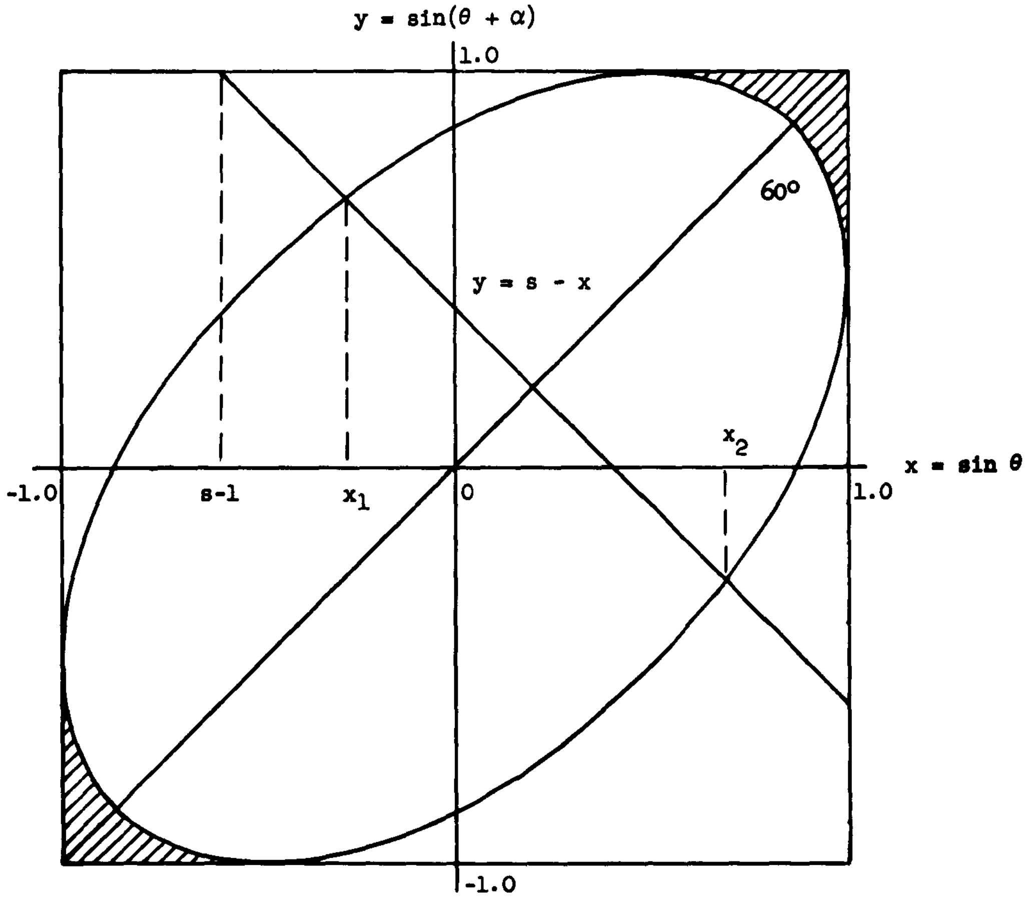
Substituting (IV-3) and (IV-5) into (IV-4), we get

$$W_2(x, y) = \frac{1}{2\Delta\pi} \operatorname{rect}\left(\frac{\sin^{-1}x}{\pi}\right) \operatorname{rect}\left(\frac{\sin^{-1}y - \sin^{-1}x}{2\alpha}\right) \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \quad . \qquad \text{(IV-6)}$$

The general outlines of $W_2(\theta, \theta + \alpha)$ and $W_2(x, y)$ in their respective planes are shown in Fig. 24. The altitude of $W_2(\theta, \theta + \alpha)$ is constant over the central area within the boundaries, but is doubled over the shaded portions at two corners due to the repetition property of $\sin(\theta + \alpha)$ when $\theta + \alpha > \frac{\pi}{2}$. The function $W_2(x, y)$ is not constant, but is contained within the ellipse. Over the shaded area at two corners the function is doubly folded for the above mentioned reason.



(a) Outlines of $W_2(\theta, \theta + \alpha)$,
θ and α are independent.



(b) Outlines of $W_2(x, y)$, $x = \sin\theta$, $y = \sin(\theta + \alpha)$.
Fig. 24.

Sum

Now we may proceed to derive the p.d.f. of the sum p(s). With the help of Fig. 24 it is evident that

$$p(s) = \int_{s-1}^{1} W_2(x,\ y = s-x)\ dx$$

$$= \frac{1}{2\Delta\pi} \int_{s-1}^{1} \mathrm{rect}\left(\frac{\sin^{-1}x}{\pi}\right) \mathrm{rect}\left(\frac{\sin^{-1}(s-x) - \sin^{-1}x}{2\Delta}\right) \frac{1}{\sqrt{1-x^2}\sqrt{1-(s-x)^2}}\ dx \quad . \tag{IV-7}$$

Within the upper and lower integration limit of x, the function

$$\mathrm{rect}\left(\frac{\sin^{-1}x}{\pi}\right) = 1 \quad . \tag{IV-8}$$

However, in the same range

$$\mathrm{rect}\left(\frac{\sin^{-1}(s-x) - \sin^{-1}x}{2\Delta}\right) = \begin{cases} 0 & s-1 < x < x_1 \\ 1 & x_1 < x < x_2 \\ 0 & x_2 < x \end{cases} \quad , \tag{IV-9}$$

where x_1 and x_2 are the roots of the equation

$$\sin^{-1}(s-x) - \sin^{-1} x = \Delta \quad . \tag{IV-10}$$

Therefore, equation (IV-7) may be rewritten as

$$p(s) = \frac{1}{2\Delta\pi} \int_{x_1}^{x_2} \frac{1}{(1-x^2)\ [1-(s-x)^2]}\ dx \quad . \tag{IV-11}$$

This equation is applicable for the range of s

$$0 < s < 1 + a \qquad \text{where} \qquad a = \cos\Delta \quad . \tag{IV-12}$$

For the range

$$1 + a < s < \sqrt{2(1 + a)}$$

the line corresponding to

$$x + y = s$$

cuts partly through the ellipse and partly through the shaded region of the two corners. Since the integrands are doubled in two of the three segments of the line, the integral of (IV-7) is better expressed in three parts:

$$p(s) = \frac{1}{2\Delta\pi}\left[2\int_{s-1}^{x_1} \quad + \int_{x_1}^{x_2} \quad + 2\int_{x_2}^{1}\right] \quad . \tag{IV-13}$$

The two roots x_1 and x_2 of equation (IV-10) are identical when

$$s = \sqrt{2(1 + a)} = s_{max} \tag{IV-14}$$

which corresponds to the case when the line $x + y = s$ is just tangent to the ellipse of Fig. 24. Therefore, in the range

$$\sqrt{2(1 + a)} < s < 2 \tag{IV-15}$$

the required p.d.f. is

$$p(s) = \frac{1}{2\Delta\pi} \cdot 2\int_{s-1}^{1} \frac{1}{\sqrt{(1-x^2)\,[1-(s-x)^2]}}\,dx \quad . \tag{IV-16}$$

The Integrals in (IV-11), (IV-13) and (IV-16) are all reducible to the elliptic integrals of the first kind. See, for example, p. 71 of Peirce, "A Short Table of Integrals." The resulting expressions are summarized on the next page:

(1) $0 < s < 1 + a \qquad (a = \cos \Delta)$

$$p(s) = \frac{1}{2\Delta\pi} [F(\alpha, \phi_1) - F(\alpha, \phi_2)]$$

where F is the elliptic integral of the first kind,

$$\alpha = \sin^{-1} k, \qquad k = \sqrt{(1 + \tfrac{s}{2})(1 - \tfrac{s}{2})}$$

$$\phi_1 = \sin^{-1} b, \qquad \phi_2 = \sin^{-1} b_2 \tag{IV-17}$$

$$b_1 = \sqrt{\left(\frac{2}{2-s}\right)\left(\frac{1-x_1}{s + 1 - x_1}\right)}, \qquad b_2 = \sqrt{\left(\frac{2}{2-s}\right)\left(\frac{1-x_2}{s + 1 - x_2}\right)}$$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} = \frac{s}{2}\left[1 \mp \sqrt{(1 - a)\left(\frac{2}{s^2} - \frac{1}{1 + a}\right)}\right]$$

(2) $1 + a < s < \sqrt{2(1 + a)}$

$$p(s) = \frac{1}{2\Delta\pi}\left[F(\alpha, \phi_1) + 3F(\alpha, \phi_2)\right] \tag{IV-18}$$

where the symbols are the same as in (1)

(3) $2(1 + a) < s < 2$

$$p(s) = \frac{1}{\Delta\pi} K(\alpha) \tag{IV-19}$$

where K is the complete elliptic integral of the first kind.

The graphs of $p(s)$ are plotted in Fig. 25 for 3 values of $a(= \cos \Delta)$. The corresponding outlines of $W_2(x, y)$ are shown in Fig. 26. The values of s_{max} are given by equation (IV-14). It is seen that for $a = 1$, i.e., $\Delta = 0$, x and y are entirely dependent and $y = x$ with probability one. Therefore, $s = 2x$, and $p(s)$ is similar to $p_1(x)$ except for a scale factor of 2. For $a = -1$, i.e., $\Delta = \pi$, x and y are statistically independent, and the resultant $p(s)$ is identical to case 4 of section I.

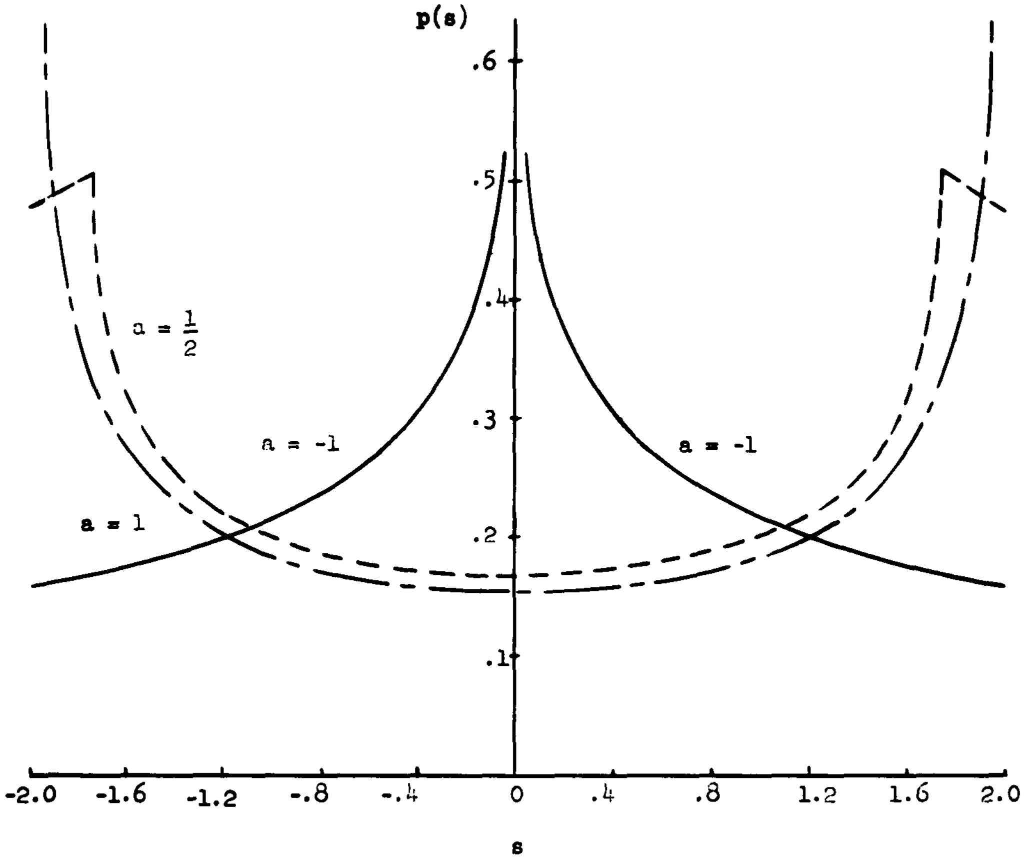
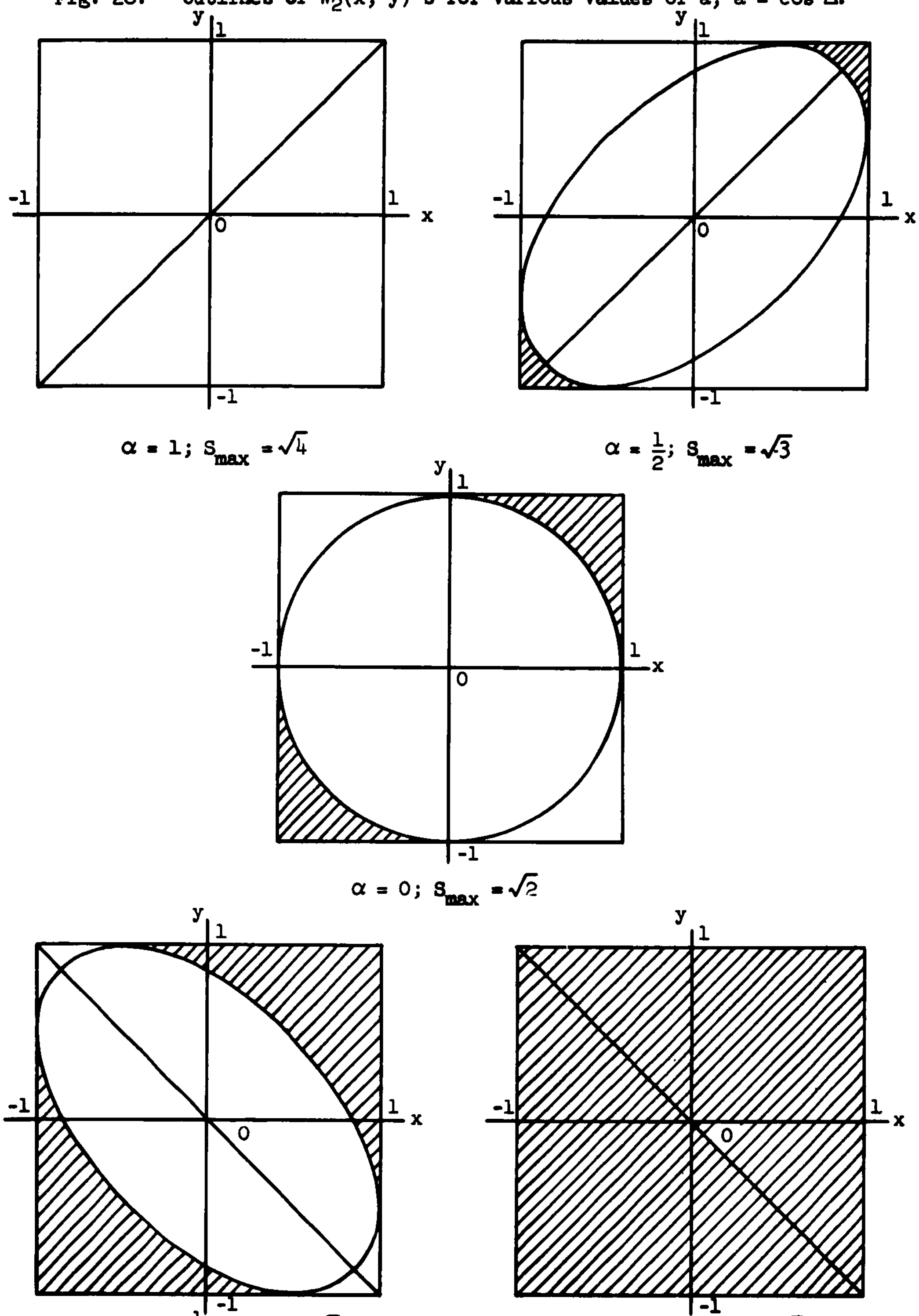


Fig. 25. P.D.F. of the Sum of Two Dependent Sinusoidal Variates.

(a = cos Δ)

Fig. 26. Outlines of $W_2(x, y)$'s for various values of a, a = cos Δ.



$\alpha = 1;\ S_{max} = \sqrt{4}$

$\alpha = \frac{1}{2};\ S_{max} = \sqrt{3}$

$\alpha = 0;\ S_{max} = \sqrt{2}$

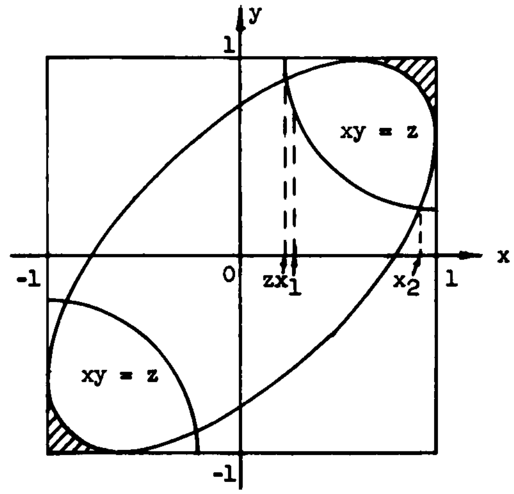
$\alpha = -\frac{1}{2};\ S_{max} = \sqrt{1}$

$\alpha = -1;\ S_{max} = \sqrt{0}$

Product

Now we proceed to derive in a similar manner the p.d.f. of the product $p(z)$, where $z = xy$ and $W_2(x, y)$ is given by equation (IV-6). Referring again to Fig. 24 the elliptical outline for $W_2(x, y)$ remains the same. However, the lines which correspond to the constant product are the hyperbolic segments

$$xy = z \tag{IV-18}$$



contained within the square of xy plane. It can be shown that the general expression of the p.d.f. of z is

$$p(z) = \frac{1}{\Delta\pi} \int_z^1 \text{rect}\left(\frac{\sin^{-1}x}{\pi}\right) \text{rect}\left[\frac{\sin^{-1}(\frac{z}{x}) - \sin^{-1}x}{2\Delta}\right] \frac{1}{\sqrt{1-x^2}\sqrt{x^2 - z^2}}\, dx \quad . \tag{IV-19}$$

Here, as before, the value of the function

$$\text{rect}\left(\frac{\sin^{-1}x}{\pi}\right) = 1$$

within the range $z < x < 1$. However, the function

$$\text{rect}\left[\frac{\sin^{-1}(\frac{z}{x}) - \sin^{-1}x}{2\Delta}\right] = 1$$

within the range $z < x_1 < x < x_2 < 1$ where x_1 and x_2 are the roots of the equation

$$\sin^{-1}\frac{z}{x} - \sin^{-1} x = \Delta \tag{IV-20}$$

Therefore, equation (IV-19) is reduced to

$$p(z) = \frac{1}{\Delta\pi}\int_{x_1}^{x_2} \frac{1}{\sqrt{1-x^2}\,\sqrt{x^2 - z^2}}\,dx \quad . \tag{IV-21}$$

This integral is again reducible to the elliptic integral of the first kind. The results are summarized below:

(1) $0 < z < a \quad (= \cos\Delta)$

$p(z) = \frac{1}{\Delta\pi}\left[F(\alpha, \phi_1) - F(\alpha, \phi_2)\right]$

where F is the elliptic integral of the first kind,

$$\begin{aligned}
&\alpha = \sin^{-1}k, && k = \sqrt{1-z^2} \\
&\phi_1 = \sin^{-1}b_1, && \phi_2 = \sin^{-1}b_2 \\
&b_1 = \frac{\sqrt{1-x_1^2}}{\sqrt{1-z_2}}, && b_2 = \frac{\sqrt{1-x_2^2}}{\sqrt{1-z^2}} \\
&\begin{matrix} x_1^2 \\ x_2^2 \end{matrix} = \frac{1 + 2az - a^2}{2} \mp \frac{1}{2}\sqrt{(1-a^2)[1 - a^2 + 4z\,(a-z)]}
\end{aligned} \tag{IV-22}$$

(2) $a < z < \frac{1 + a}{2}$

$$p(z) = \frac{1}{\Delta\pi}\left[2K(\alpha) - F(\alpha, \phi_1) + F(\alpha, \phi_2)\right] \tag{IV-23}$$

where K is the complete elliptic integral of the first kind and other symbols are the same as in (1).

(3) $\frac{1 + a}{2} < z < 1$

$$p(z) = \frac{2}{\Delta\pi} K(\alpha) \tag{IV-24}$$

(4) $-1 < z < \frac{a - 1}{2}$

$$p(z) = 0 \tag{IV-25}$$

(5) $\frac{a - 1}{2} < z < 0$

$$p(z) = \frac{1}{\Delta\pi}\left[F(\alpha, \phi_1) - F(\alpha, \phi_2)\right] \tag{IV-26}$$

The graphs of p(z) are plotted in Fig. 27 for 3 values of $a(= \cos \Delta)$. Again the cases $a = 1$ and $a = -1$ correspond to the complete dependence and independence of the variables x and y.

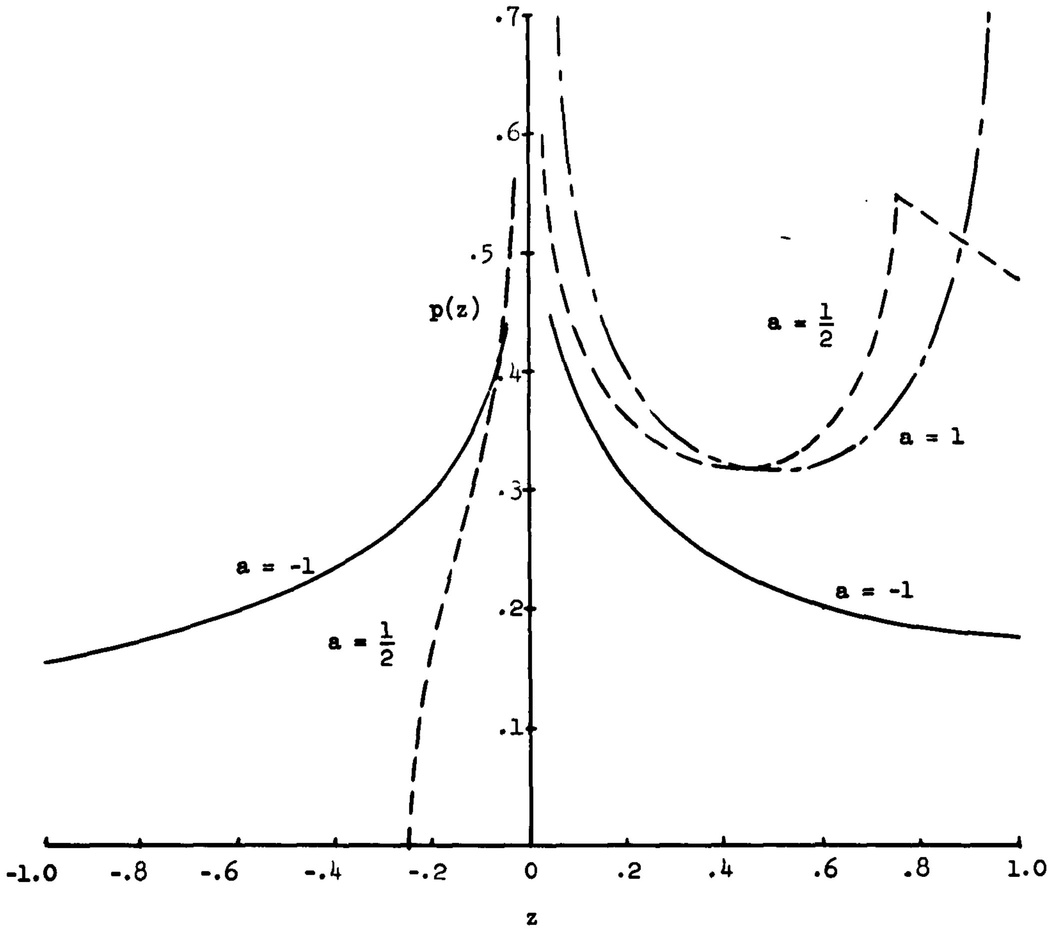


Fig. 27. P. D. F. of the Product of Two Dependent Sinusoidal Variates.

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